Friday, October 30, 2015

539: 57, 58, 61, 62, 68, 72, 73, 74

Problem 57

Problem. The surface of a machine part is the region between the graphs of $y = |x|$ and $x^2 + (y - k)^2 = 25$.

(a) Find k when the circle is tangent to the graph of $y = |x|$.

(b) Find the area of the surface of the machine part.

(c) Find the area of the surface of the machine part as a function of the radius r of the circle.

Solution. (a) The figure shows a square, which has a side of 5 (the radius). Therefore, the diagonal of the square is $5\sqrt{2}$, which is k. So $k = 5\sqrt{2}$.

(b) The equation of the circle is $x^2 + (y - 5\sqrt{2})^2 = 25$, so the equation of the bottom half is

$$y = 5\sqrt{2} - \sqrt{25 - x^2}.$$

The area of the right half of the surface is

$$\begin{aligned} A &= \int_0^{5/\sqrt{2}} \left(\left(5\sqrt{2} - \sqrt{25 - x^2} \right) - x \right) \, dx \\ &= \left[5\sqrt{2}x - \frac{25}{2} \arcsin \frac{x}{5} - \frac{1}{2} x\sqrt{25 - x^2} - \frac{1}{2}x^2 \right]_0^{5/\sqrt{2}} \\ &= 25 - \frac{25}{2} \arcsin \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{5}{\sqrt{2}} \sqrt{25 - \frac{25}{2}} - \frac{25}{4} \\ &= \frac{25}{2} - \frac{25}{2} \frac{\pi}{4} \\ &= \frac{25}{2} \left(1 - \frac{\pi}{4} \right). \end{aligned}$$

So the total area is

$$25 \left(1 - \frac{\pi}{4} \right).$$

Problem 58

Problem. The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 meter and 3 meters, respectively.

(a) Determine the volume of fluid in the tank as a function of its depth d.

(b) Graph the function in part (a).

(c) Disign a dipstick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

(d) Fluid is entering the tank at a rate of $\frac{1}{4}$ cubic meter per second. Determine the rate of change of the depth of the fluid as a function of its depth d.

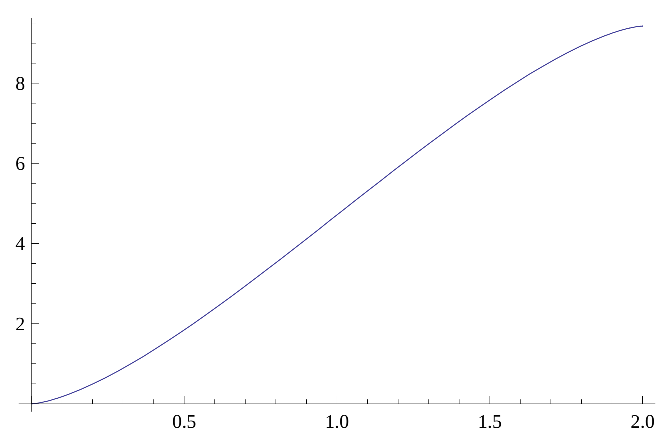
(e) Graph the function in part (d). When will the rate of change of the depth be a minimum?

Solution. (a) According to Exercise 56, the area of the cross-section (with $h = 1 - d$) is

$$\text{Area} = \frac{\pi}{2} - \arcsin(1-d) - (1-d)\sqrt{2d - d^2}.$$

So the volume of the fluid with depth d is

$$\text{Volume} = 3\left(\frac{\pi}{2} - \arcsin(1-d) - (1-d)\sqrt{2d - d^2}\right).$$

(b) 

(c) We need to solve the equation

$$\frac{\pi}{2} - \arcsin(1-d) - (1-d)\sqrt{2d - d^2} = h$$

for $h = \frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$. When $h = \frac{1}{2}$, it is clear that $d = 1$. And by symmetry, if we solve it for $h = \frac{1}{4}$, that will give us the solution for $h = \frac{3}{4}$ (when the tank is 3/4 full, it is also 1/4 empty).

Using numerical methods, we find that the tank is $\frac{1}{4}$ full when $d = 0.596$ meters. Therefore, our dipstick has marks at 0.596, 1.0, and 1.404.

(d) Differentiate the equation

$$V = 3\left(\frac{\pi}{2} - \arcsin(1-d) - (1-d)\sqrt{2d-d^2}\right)$$

with respect to time t and get

$$\begin{aligned}\frac{dV}{dt} &= 3\left(\frac{1}{\sqrt{2d-d^2}} + \sqrt{2d-d^2} - (1-d)\cdot\frac{1-d}{\sqrt{2d-d^2}}\right)\frac{dd}{dt} \\ &= 6\sqrt{2d-d^2}\left(\frac{dd}{dt}\right).\end{aligned}$$

We are given that $\frac{dV}{dt} = \frac{1}{4}$, so

$$\begin{aligned}\frac{1}{4} &= 6\sqrt{2d-d^2}\left(\frac{dd}{dt}\right), \\ \frac{dd}{dt} &= \frac{1}{24\sqrt{2d-d^2}}.\end{aligned}$$

(e) To minimize $\frac{dd}{dt}$, we must take its derivative, set it equal to 0, and solve for d.

$$\frac{d^2d}{dt^2} = -\frac{1}{24}\cdot\frac{1-d}{(2d-d^2)^{3/2}}.$$

Clear, the solution to

$$\frac{d^2d}{dt^2} = 0$$

is $d = 1$, when the tank is half full.

Problem 61

Problem.

Solution.

Problem 63

Problem.

Solution.

Problem 68

Problem.

Solution.

Problem 72

Problem.

Solution.

Problem 73

Problem.

Solution.

Problem 74

Problem.

Solution.